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LETTER TO THE EDITOR

Extending additivity from symmetric to asymmetric channels

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Abstract
We prove a lemma which allows one to extend results about the additivity of the minimal output entropy from highly symmetric channels to a much larger class. A similar result holds for the maximal output p-norm. Examples are given showing its use in a variety of situations. In particular, we prove the additivity and the multiplicativity for the shifted depolarizing channel.

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A natural and important class of measures of the noisiness of a quantum channel are based on optimal output purity, i.e., how close can the output be to a pure state as measured by the minimal output entropy (MOE) or the maximal output p-norm. It is then natural to ask if a tensor product of two channels can ever be less noisy in the sense that some entangled input can have its output closer to a pure state than the product of the optimal inputs of the two channels. This leads to the conjectures of the additivity of the MOE and the multiplicativity of the maximal output p-norm. Although the additivity of the MOE has been shown [1] to be equivalent globally to several other fundamental conjectures in quantum information theory; the additivity of the Holevo capacity, the additivity of the entanglement of formation and the strong superadditivity of the entanglement of formation, we consider only the output purity.

Intuitively, one would expect that entanglement would be more likely to enhance output purity for highly symmetric channels than for non-unital or asymmetric ones. Symmetric channels allow one to construct maximally entangled states as superpositions of products of optimal inputs, but this cannot be done for a channel with a unique optimal input. Curiously, however, most channels for which the additivity or the multiplicativity has been proven are highly symmetric: unital qubit channels [2], the depolarizing channel [3, 4], the Werner–Holevo channel [5–7], and the transpose depolarizing channel [8, 9]. These proofs exploit the symmetric structures of these channels. By contrast, asymmetric channels have been extremely resistant to proofs. Indeed, apart from entanglement-breaking channels [10] and one modification of the Werner–Holevo channel [11], the additivity has not been proven for any non-unital channels and the multiplicativity only at $p = 2$ in a few additional cases [12, 13].

In this letter, we present a lemma which allows one to prove the additivity of the MOE and the multiplicativity of the maximal output p-norm for a large class of channels. These include the shifted depolarizing channel introduced in [12], as well as many other types of non-unital channels for which our methods can show that multiplicativity holds for values of p which include the interval [1, 2]. In essence, the lemma allows the extension of the additivity and the multiplicativity from highly symmetric channels to asymmetric ones.

Let us give some basic definitions. A quantum state is represented as a positive semidefinite operator ρ of trace one in a Hilbert space $\mathcal{H}$ of dimension d; this is called a density operator. The set of density operators in $\mathcal{H}$ is written as $\mathcal{D}(\mathcal{H})$. A channel Φ on $\mathcal{H}$ is a completely positive trace-preserving (CPTP) map acting on $\mathcal{D}(\mathcal{H})$. A channel Φ is called unitarily covariant if for any unitary operator U there exists a unitary operator V such that

$$\Phi(U\rho U^*) = V\Phi(\rho)V^* \tag{1}$$

for $\forall\, \rho \in \mathcal{D}(\mathcal{H})$. The MOE of Φ is defined as

$$S_{\min}(\Phi) := \inf_{\rho\in\mathcal{D}(\mathcal{H})} S(\Phi(\rho)), \tag{2}$$

where S is the von Neumann entropy: $S(\rho) = -\mathrm{tr}[\rho \log \rho]$. The maximal output p-norm of Φ is defined as

$$\nu_p(\Phi) := \sup_{\rho\in\mathcal{D}(\mathcal{H})} \|\Phi(\rho)\|_p, \tag{3}$$

where $\| \, \|_p$ is the Schatten p-norm: $\|\rho\|_p = (\mathrm{tr}|\rho|^p)^{\frac{1}{p}}$.

The additivity conjecture of the MOE is that for channels Φ and Ω

$$S_{\min}(\Phi\otimes\Omega) = S_{\min}(\Phi) + S_{\min}(\Omega). \tag{4}$$

Note that the bound $S_{\min}(\Phi\otimes\Omega) \leqslant S_{\min}(\Phi) + S_{\min}(\Omega)$ is straightforward. The multiplicativity conjecture is that for channels Φ and Ω

$$\nu_p(\Phi\otimes\Omega) = \nu_p(\Phi)\nu_p(\Omega), \tag{5}$$

for any $p \in [1, 2]$. Note that the bound $\nu_p(\Phi\otimes\Omega) \geqslant \nu_p(\Phi)\nu_p(\Omega)$ is straightforward. Note also that if there exists a sequence $\{p_n\}$ with $p_n \searrow 1$ for which the multiplicativity (5) holds then the additivity of the MOE (4) also holds [14].

Lemma 1. *Suppose we have two channels on $\mathcal{H}$: a unitarily covariant channel Ψ and a channel M such that $M(\rho_0)$ is of rank one for some state $\rho_0 \in \mathcal{D}(\mathcal{H})$. Take another Hilbert space $\mathcal{K}$ and a channel Ω on $\mathcal{K}$.*

(1) If the maximal output p-norm of $\Psi\otimes\Omega$ is multiplicative then so is that of $(\Psi\circ M)\otimes\Omega$ (for the same p).
(2) If the MOE of $\Psi\otimes\Omega$ is additive then so is that of $(\Psi\circ M)\otimes\Omega$.

Proof. We first claim that the maximal output p-norm of $\Psi\circ M$ satisfies

$$\begin{aligned}\nu_p(\Psi\circ M) &= \sup_{\rho\in\mathcal{D}(\mathcal{H})} \|\Psi(M(\rho))\|_p \\ &= \sup_{\sigma\in M(\mathcal{D}(\mathcal{H}))} \|\Psi(\sigma)\|_p = \nu_p(\Psi).\end{aligned} \tag{6}$$

Since Ψ is unitarily covariant the maximal output p-norm of Ψ is attained at any state of rank one. On the other hand, $M(\mathcal{D}(\mathcal{H}))$, the image of $\mathcal{D}(\mathcal{H})$ by M, has such a state $M(\rho_0)$ for some

$\rho_0 \in \mathcal{D}(\mathcal{H})$. This verifies (6). Next, take any state $\hat{\rho} \in \mathcal{D}(\mathcal{H} \otimes \mathcal{K})$ then

$$\begin{aligned} \|((\Psi \circ M) \otimes \Omega)(\hat{\rho})\|_p &= \|(\Psi \otimes \Omega)((M \otimes \mathbf{1}_{\mathcal{K}})(\hat{\rho}))\|_p \\ &\leqslant \nu_p(\Psi)\nu_p(\Omega) = \nu_p(\Psi \circ M)\nu_p(\Omega). \end{aligned} \tag{7}$$

Therefore we have

$$\nu_p((\Psi \circ M) \otimes \Omega) \leqslant \nu_p(\Psi \circ M)\nu_p(\Omega). \tag{8}$$

Similarly, for any state $\hat{\rho} \in \mathcal{D}(\mathcal{H} \otimes \mathcal{K})$

$$\begin{aligned} S(((\Psi \circ M) \otimes \Omega)(\hat{\rho})) &\geqslant S_{\min}(\Psi) + S_{\min}(\Omega) \\ &= S_{\min}(\Psi \circ M) + S_{\min}(\Omega). \end{aligned} \tag{9}$$

Hence we have

$$S_{\min}((\Psi \circ M) \otimes \Omega) \geqslant S_{\min}(\Psi \circ M) + S_{\min}(\Omega). \tag{10}$$

Combining these inequalities with those noted above in the reverse direction completes the proof. □

Remark. The conditions on Ψ and M can be weakened to verify that a maximizer of $\|\Psi(\rho)\|_p$ (or a minimizer of $S(\Psi(\rho))$) coincides with an output of M.

We now use the lemma to obtain some new results. Our first group of examples are based on the depolarizing channel defined by

$$\Delta_\lambda^{(d)}(\rho) = \lambda\rho + (1-\lambda)\frac{1}{d}I. \tag{11}$$

Here $\lambda \in [-1/(d^2-1), 1]$, d is the dimension of the signal Hilbert space $\mathcal{H}$ and I is the identity operator. The additivity of the MOE (4) and the multiplicativity (5) with $p \in [1, \infty]$ have been proven for $\Delta_\lambda^{(d)} \otimes \Omega$ with Ω arbitrary [3]. When M is a channel with an output of rank one, consider the class of channels:

$$\Phi(\rho) = \left(\Delta_\lambda^{(d)} \circ M\right)(\rho) = \lambda M(\rho) + (1-\lambda)\frac{1}{d}I. \tag{12}$$

Since the depolarizing channel is unitarily covariant, the additivity of the MOE (4) and the multiplicativity (5) for $p \in [1, \infty]$ hold for the product of a channel of the form (12) and an arbitrary channel. We discuss examples of this class below.

Example 1. The shifted depolarizing channel is defined by

$$\Phi(\rho) = a\rho + b|\phi\rangle\langle\phi| + c\frac{1}{d}I. \tag{13}$$

Here $|\phi\rangle\langle\phi|$ is a fixed state of rank one, $a, b, c \geqslant 0$ and $a+b+c=1$. The multiplicativity (5) for $p = 2$ was proven for the shifted depolarizing channel [12, 13]. To see that this channel has the form (12), define a channel M by

$$M(\rho) = \frac{a}{a+b}\rho + \frac{b}{a+b}|\phi\rangle\langle\phi|, \tag{14}$$

and put $\lambda = a + b$.

Example 2. Let M be a channel of the form:

$$M(\rho) = \frac{1}{\lambda}\sum_k \lambda_k V_k \rho V_k^*. \tag{15}$$

Here $\lambda_k > 0, \lambda = \sum_k \lambda_k \leqslant 1$, and V_k are unitary operators having a common eigenvector. Then define a channel

$$\Phi(\rho) = \sum_k \lambda_k V_k \rho V_k^* + (1-\lambda)\frac{1}{d} I. \tag{16}$$

This channel was introduced in [15], where both the additivity of the MOE (4) and the multiplicativity (5) were proven by a different method.

Next we consider qubit channels. The additivity of the MOE (4) and the multiplicativity (5) were proven for unital qubit channels [2]. We use our results to extend this to some non-unital qubit channels. Recall that any qubit state can be written as

$$\rho = \frac{1}{2}\left[I + \sum_{k=1}^{3} w_k \sigma_k\right]. \tag{17}$$

Here $\mathbf{w} \in \mathbf{R}^3$ with $|\mathbf{w}| \leqslant 1$ and σ_k are pauli matrices. Note that ρ is of rank one if and only if $|\mathbf{w}| = 1$. Let $\Upsilon_{\mathbf{x},t}$ denote the channel:

$$\Upsilon_{\mathbf{x},t}(\rho) = \tfrac{1}{2}[I + x_1 w_1 \sigma_1 + x_2 w_2 \sigma_2 + (t + x_3 w_3)\sigma_3]. \tag{18}$$

Here we assume for simplicity $x_k > 0$ for $k = 1, 2, 3$ and $t > 0$. Then we need the conditions $x_1, x_2, x_3 + t \leqslant 1$, and

$$(x_1 \pm x_2)^2 \leqslant (1 \pm x_3)^2 - t^2. \tag{19}$$

These conditions are necessary and sufficient for a map of the form (18) to be CPTP [16, 17]. The qubit depolarizing channel $\Delta_\lambda^{(2)}$ belongs to this class with $x_1 = x_2 = x_3 = \lambda$ and $t = 0$. In the Bloch sphere representation (17) the image of the sphere by a channel of the form (18) is an ellipsoid with axes of length x_k and centre $(0, 0, t)$.

Example 3. Let $M = \Upsilon_{\mathbf{x},t}$ with

$$x_3 = x_1 x_2, \qquad t^2 = \left(1 - x_1^2\right)\left(1 - x_2^2\right) \neq 0 \tag{20}$$

then M is an extreme point of the set of qubit CPTP maps and has an output of rank one. In fact, it has two such outputs [16] unless $x_1 = \pm x_2$, when it reduces to the amplitude damping channel. In both cases the lemma can be applied to the channel,

$$\Upsilon_{\mathbf{y},u}(\rho) = \left(\Delta_\lambda^{(2)} \circ M\right)(\rho) = \tfrac{1}{2}[I + \lambda x_1 w_1 \sigma_1 + \lambda x_2 w_2 \sigma_2 + \lambda(t + x_3 w_3)\sigma_3], \tag{21}$$

to show that the additivity of the MOE (4) and the multiplicativity (5) for $p \in [1, \infty]$ hold for $\Upsilon_{\mathbf{y},u} \otimes \Omega$ with Ω arbitrary. Here the conditions (20) imply $y_1 y_2 = \lambda y_3$, hence $|y_1 y_2| \leqslant |y_3|$ and $u^2 = \left(y_2^2 - y_3^2\right)\left(y_1^2 - y_3^2\right)/y_3^2$. In this example the shift $(0, 0, u)$ is in the direction of the shortest axes of the ellipsoid.

Example 4. Let $M = \Upsilon_{\mathbf{x},1-x_3}$, in which case the constraints (19) are equivalent to

$$x_1 = x_2, \qquad x_i^2 \leqslant x_3 \qquad (i = 1, 2). \tag{22}$$

Then $\frac{1}{2}[I + \sigma_3]$ is a stationary point so that the channel has an output of rank one. So the channel,

$$\Upsilon_{\mathbf{y},u}(\rho) = \left(\Delta_\lambda^{(2)} \circ M\right)(\rho) = \tfrac{1}{2}[I + \lambda x_1 w_1 \sigma_1 + \lambda x_2 w_2 \sigma_2 + \lambda(1 - x_3 + x_3 w_3)\sigma_3], \tag{23}$$

satisfies the additivity of the MOE (4) and the multiplicativity (5) for $p \in [1, \infty]$ with Ω arbitrary. Note that (22) implies that $y_1 = y_2$ and

$$y_i^2 \leqslant y_3(y_3 + u) \qquad (i = 1, 2), \tag{24}$$

and that, conversely, any channel of the form (18) satisfying (24) and $y_1 = y_2$ can be written in the form (23). In this case (24) is slightly weaker than requiring that the shift $(0, 0, u)$ is in the direction of the longest axes of the ellipsoid.

Example 5. In this example we will apply the remark following the lemma rather than the lemma itself. Let $\Psi_{\lambda_k} = \Upsilon_{\mathbf{z},0}$ be the unital qubit channel with $z_k = \lambda_k$, and let M be as in the previous example. With the additional assumption,

$$|\lambda_i| \leqslant |\lambda_3| \qquad (i = 1, 2) \tag{25}$$

the optimal inputs of Ψ_{λ_k} are $\frac{1}{2}[I \pm \sigma_3]$. Therefore, Ψ_{λ_k} and M satisfy the conditions in the remark so that the channel,

$$\begin{aligned} \Upsilon_{\mathbf{y},u}(\rho) &= (\Psi_{\lambda_k} \circ M)(\rho) \\ &= \tfrac{1}{2}[I + \lambda_1 x_1 w_1 \sigma_1 + \lambda_2 x_2 w_2 \sigma_2 + \lambda_3 (1 - x_3 + x_3 w_3)\sigma_3], \end{aligned} \tag{26}$$

satisfies the additivity of the MOE (4) and the multiplicativity (5) for $p \in [1, \infty]$ with Ω arbitrary. Note that $\Upsilon_{\mathbf{y},u}$ has the form (18) with parameters which satisfy (24) and

$$(y_1 - y_2)^2 \leqslant \frac{y_3}{y_3 + u}(1 - y_3 - u)^2. \tag{27}$$

Conversely, any channel of the form (18) which satisfies the conditions (24) and (27), which are stronger than (19), can be written in the form (26) with parameters of the channel $M = \Upsilon_{\mathbf{x},1-x_3}$ chosen so that $x_1^2 = x_2^2 = x_3$. This example is of most interest when $y_1 \neq y_2$ since it cannot be written in the form (23). A channel with $y_1 = y_2$ satisfying (24) and (27) with strict inequality can be written in the form (23) with $x_i^2 < x_3$ or in the form (26) with $x_i^2 = x_3$ and $0 < \lambda_i < \lambda_3$ for $i = 1, 2$.

Finally let us remark that one can compose these M with other unitarily covariant channels Ψ such as the transpose depolarizing channel to get channels:

$$\Phi(\rho) = (\Psi \circ M)(\rho) = \lambda M(\rho)^{\mathrm{T}} + (1 - \lambda)\frac{1}{d}I. \tag{28}$$

Here $\lambda \in [-1/(d-1), 1/(d+1)]$, T is the transpose. Note that the additivity of the MOE (4) for the transpose depolarizing channel was proven [8, 9]. In addition, composition with the Werner–Holevo channel yields another class of channels of the form $(I - [M(\rho)]^T)/(d-1)$. The additivity of the MOE (4) and the multiplicativity (5) of these channels were proven in [11] by a different method. The channel M given by (14) in this example was called 'stretching' there.

We have shown how to extend the known results about the additivity of the MOE and the multiplicativity of the maximal output p-norm to a much larger class of channels, including many non-unital and asymmetric ones.

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References

[1] Shor P W 2004 *Commun. Math. Phys.* **246** 453–72

[2] King C 2002 *J. Math. Phys.* **43** 4641–53

[3] King C 2003 *IEEE Trans. Inf. Theory* **49** 221–9

[4] Fujiwara A and Hashizumé T 2002 *Phys. lett.* A **299** 469–75

[5] Matsumoto K and Yura F 2004 *J. Phys. A: Math. Gen.* **37** L167–71

[6] Datta N, Holevo A S and Suhov Y 2004 A quantum channel with additive minimum output entropy *Preprint* quant-ph/0403072

[7] Alicki R and Fannes M 2004 Note on multiple additivity of minimal Renyi entropy output of the Werner–Holevo channels *Preprint* quant-ph/0407033

[8] Fannes M, Haegeman B, Mosonyi M and Vanpeteghem D 2004 Additivity of minimal entropy output for a class of covariant channels *Preprint* quant-ph/0410195

[9] Datta N, Holevo A S and Suhov Y 2004 Additivity for transpose depolarizing channels *Preprint* quant-ph/0412034

[10] Shor P W 2002 *J. Math. Phys.* **43** 4334–40

[11] Wolf M M and Eisert J 2005 *New J. Phys.* **7** 93

[12] Giovannetti V, Lloyd S and Ruskai M B 2005 *J. Math. Phys.* **46** 042105

[13] King C, Nathanson M and Ruskai M B 2004 Multiplicativity properties of entrywise positive maps *Preprint* quant-ph/0409181

[14] Amosov G G, Holevo A S and Werner R F 2000 *Problems Inf. Transm.* **36** 305–13

[15] Datta N and Ruskai M B 2005 Maximal output purity and capacity for asymmetric unital qudit channels *J. Phys. A: Math. Gen.* **38** 9785 (*Preprint* quant-ph/0505048)

[16] Ruskai M B, Szarek S and Werner E 2002 *Linear Algebr. Appl.* **347** 159–87

[17] Fujiwara A and Algoet P 1999 *Phys. Rev.* A **59** 3290–4